Tying Topology Tools Together

and geometry, algebra, and more

Rob Scharein

Centre for Experimental and
Constructive Mathematics / CoLab
Simon Fraser University

New Media Innovation Centre

WestGrid

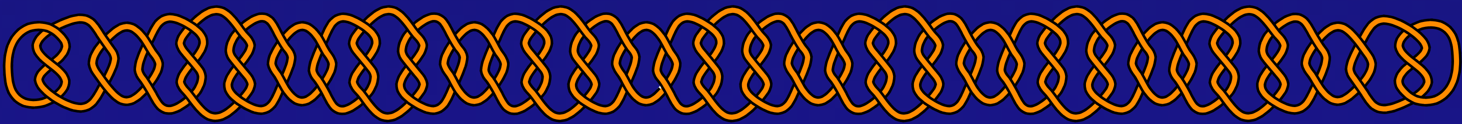
Media and Graphics Interdisciplinary Centre
University of British Columbia

ICIAM2003, Sydney
11 July 2003

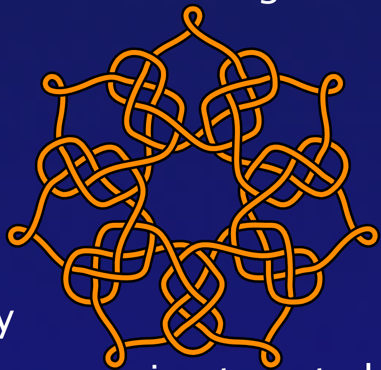
Introduction

- Traditional drawing packages (Gnuplot, Photoshop, paint programs) are often inadequate for *correctly* drawing or manipulating mathematical objects.
- There do exist excellent *geometric drawing* tools (Geometer's Sketchpad, Cinderella, and others).
- These are useful tools for *mathematical drawing*, in the sense of Roseman (1992).
- However, none of these tools are really designed for the flexible world of topology.

Therefore we have choosen one area of three-dimenional topology, *knot theory*, as an arena for exploring *topological drawing.*



KnotPlot is a program for visualizing, constructing and interacting with knots and links in three and four dimensions.

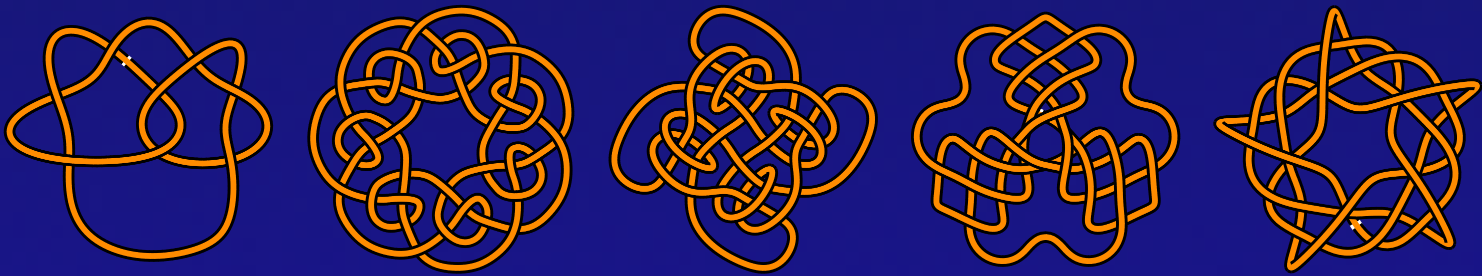


- A knot viewer
- A *topological drawing* tool (Roseman, 92)
  - correct display of mathematical objects
  - more concerned with topology than geometry
  - exact geometry often a result of automated or semi-automated relaxation algorithms
  - maintain essential topological features during interactions
  - Knots can be created from topological descriptions
- Available for most computer architectures (OpenGL/GLUT)
- Useful for both knot theorists as well as traditional knot tyers

Looking at knots

Large number of knots “built-in”

- Complete Rolfsen catalogue (384 knots and links) Knot zoo
- Specialized knots
  - composite knots
  - examples from knot theory as well as decorative knots
  - knots with specialized names when in certain configurations (*eg.*, Borromean rings)
  - “practical” knots (Ashley)
  - Celtic knots



Constructing knots

In addition to the catalogue, many classes of knots can be constructed directly

- torus knots
- knot chains
- from symbolic notation systems
  - Conway notation
  - braid word
  - Dowker code
- stack–based *tangle calculator*
- “freehand” sketching

Transforming knots

Knots may be transformed into new knots via a number of procedures

- direct manipulation (dragging)
- cutting, splicing, (the usual editing functions)
- higher level operations, such as *warping*

(Demo of constructing and transforming knots)

Topological refinement

After the embedding of a knot is specified, a user may use KnotPlot to refine that embedding in a number of ways.

- Simple force model
- Attractive "mechanical" force between adjacent vertices

$$F_e = Kr^{-(2+\alpha)} \tag{1}$$

  where $\beta = 0$ is the ideal (linear) spring

- Repulsive "electrical" force between non-adjacent pairs of vertices

$$F_e = Kr^{-(2+\alpha)} \tag{2}$$

  where $\alpha = 0$ is a Coulomb falloff $(1/r^2)$

Other force models

- damping
- related to energy models *eg.*, symmetric energy (Buck)
- “thermal” agitation
- interactive “pushing” and “pulling”
- different masses for vertices

To be a topologically valid tool, KnotPlot must ensure that it does not change the knot type during the relaxation procedure.

- each vertex is moved one at a time
- vertices move maximum distance of d_{max} each relaxation step
- if $d_{\text{max}} < d_{\text{close}}$, where d_{close} is the minimum distance between any two non–adjacent edges then the knot type is maintained
- collision checking can be turned off

This method works well in many cases.

Other tricks

- Since we aren't doing molecular biology, we can feel free to extend the "physics" of the relaxation by
  - deleting vertices when possible
  - adding extra vertices where things might be tight
- The above have to be done in a manner that preserves knot type
- Doesn't always work! Sometimes "global" operations are needed.
  - Big nasty unknot created by Ochiai
  - Software by Milana Huang can untie this knot, but only after 107 hours of clock time (on an Onyx)
  - Possible that KnotPlot would too, if given the time

Applications

KnotPlot most commonly used for mathematical illustration, not experimental knot theory.

- exports PostScript in a large number of flavours
- exports PL and DIFF representations in several 3D file formats for input into raytracers or other renderers

  Example 1 Example 2

Knot theoretical applications

- support routine tasks in knot theory
  - knot polynomial calculation
  - geometric properties: writhe, average crossing number, thickness
  - topological properties: linking number, Dowker code
- scripting language for running experiments
- interface between Maple, Mathematica, SnapPea (Weeks)

Stick number problem

What is the minimal number of sides a polygon must have to be a representative of a given knot type? (Randell, Meissen)

- Theoretical numbers known for only a few knots
- Some early experimental work done by Meissen
  - knots up to seven crossings
  - four knots of eight crossings
  - KnotPlot uses a “brute-force” approach involving random agitation, opportunistic deletion/addition of extra vertices, and ample computer time

knot	sticks
3_1	$6^\star$
4_1	$7^\star$
5_1	$8^\star$
5_2	$8^\star$
6_1	$8^\star$
6_2	$8^\star$
6_3	$8^\star$
7_1	9
7_2	9
7_3	9
7_4	9

knot	sticks
7_5	9
7_6	9
7_7	9
8_1	10
8_2	10
8_3	10
8_4	10
8_5	10
8_6	10
8_7	10
8_8	10

knot	sticks
8_9	10
8_{10}	10
8_{11}	10
8_{12}	10
8_{13}	10
8_{14}	10
8_{15}	10
8_{16}	9
8_{17}	9
8_{18}	9
8_{19}	$8^\star$

knot	sticks
8_{20}	$8^\star$
8_{21}	9
Granny	$8^\star$
Square	$8^\star$
$3_1\sharp 3_1\sharp 3_1$	11 (10)
$3_1\sharp 3_1\sharp \overline{3_1}$	11 (10)
$K_{4,5}$	$10^\star$
$K_{5,6}$	13 (12)
$K_{6,7}$	16 (14)
$K_{7,8}$	19 (16)
$K_{3,4}\sharp K_{3,4}$	13 (12)

Provisional stick numbers for knots. A * indicates that the value is known to be best value possible (*i.e.*, it is the stick number of the knot). Values in parentheses are stick numbers known from theoretical results.

Sticks	Alternating	Non-alternating	Total
9	4	5	9
10	26	3	29
11	11	0	11
Total	41	8	49

Provisional stick-number results for the nine crossing knots.

Sticks	Alternating	Non-alternating	Total
10	13	29	42
11	54	13	67
12	50	0	50
13	5	0	5
14	1	0	1
Total	123	42	165

Provisional stick-number results for the ten crossing knots.
The single “problem” knot of 14 sticks is 10_{84}.

Equilateral stick numbers

How does the stick number change if we constraint the knot to be an equilateral polygon? (work together with Eric Rawdon)

- Strangely, not much.

  (look at data)

- A few knots, such as 8_{19}, appear to genuinely have an equilateral stick number (9) greater than the unconstrained stick number (8)
- Which knots have the same stick number for both cases?
- Does this show a limitation of KnotPlot, or is it real?
- How does the fraction of exceptional knots increase (decrease?) as crossing number increases?

Many of the stick knots exhibited interesting symmetries in a minimal energy conformation, using the *minimum distance energy* of Simon.
If knot K is defined by the vertex positions $\vec{p}_0, \vec{p}_1, \ldots \vec{p}_{n-1}$, where $\vec{e}_k$ is the edge from $\vec{p}_k$ to $\vec{p}_{k+1}$ (all numbers being taken modulo n), then we have

$$E_{\mathsf{MD}}(K) = \sum_{\substack{\vec{e}_i, \vec{e}_j \text{ not} \\ \text{adjacent}}} \frac{\|\vec{e}_i\| \, \|\vec{e}_j\|}{\mathsf{MD}(\vec{e}_i, \vec{e}_j)^2} \tag{3}$$

where $\mathsf{MD}(\vec{e}_i, \vec{e}_j)$ is the minimum distance between any point on edge $\vec{e}_i$ and any point on edge $\vec{e}_j$.
This energy model has a number of interesting properties.

(look at these symmetries)

Hyperbolic knot census

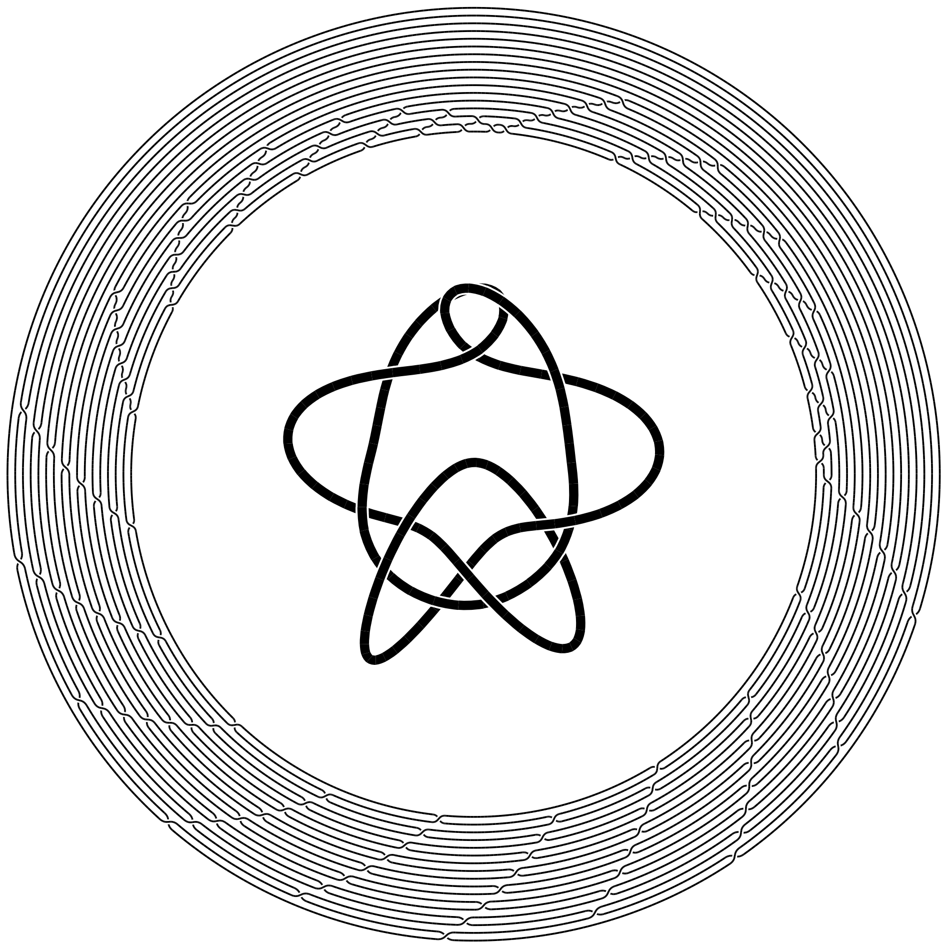
Conducted by Callahan, Dean, and Weeks to list knots according to the complexity of their complements.

- knots ordered according to how many ideal tetrahedra are in their complement
- Figure–8 knot is simplest knot in this sense, with only two tetrahedra
- found 72 knots with six or fewer tetrahedra
- complexity of complement has little to do with complexity of knot
- some knots were already known from tables, others had simple Dowker code descriptions
- other knots had complex braid descriptions

$$(\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\sigma_{10}^{-1}\sigma_{12}^{-1}\sigma_{14}^{-1}\sigma_{16}^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_9^{-1}\sigma_{11}^{-1}\sigma_{13}^{-1}\sigma_{15}^{-1})^5\times$$
$$(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1)^2$$

Complex braids could be simplified greatly

- topological refinement applied in an interactive fashion
- difficult to duplicate
- often simplification was dramatic (look at the hyperbolic knot census or show how it was done)



Closed braid with word $(\sigma_2^{-1}\sigma_4^{-1}\sigma_6^{-1}\sigma_8^{-1}\sigma_{10}^{-1}\sigma_{12}^{-1}\sigma_{14}^{-1}\sigma_{16}^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_7^{-1}\sigma_9^{-1}\sigma_{11}^{-1}\sigma_{13}^{-1}\sigma_{15}^{-1})^5 \times$ $(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1)^2$ and the knot after simplification (centre).

Visualizing the symmetric energy

The *symmetric energy* considers the knot as a radiating tube of small thickness (Buck). Let K be a smooth knot parameterized by $x(t)$ and let x, y be arbitrary points $x(t), y(t)$ in K, then

$$E_S(K) = \int\int \frac{|d\mathbf{x} \times r||d\mathbf{y} \times r|}{|x - y|^2} \tag{4}$$

where $d\mathbf{x} = \dot{x}(t)dt$ is the line element at x and $r = (x - y)/|(x - y)|$ is the unit vector in the direction of x from y.
Buck showed this is related to the *average crossing number*

$$\mathcal{ACN}(K) = \frac{1}{4\pi} \iint_{K \times K} \frac{|r \cdot d\mathbf{x} \times d\mathbf{y}|}{|x - y|^2} \tag{5}$$

by the relation

$$E_S(K) \geq 4\pi \mathcal{ACN}(K) \tag{6}$$

Using a simple raytracing technique, we can visualize the symmetric energy.

(look at some radiating tubes)

Random knots

As part of *physical knot theory*, a fair bit of work has been done with random knotting.

- Important theoretical results due to Pippenger (gaussian random knotting), Wittington (SAW on cubic lattice), Soteros, and others.
- Probability of being knotted tends to unity as length of knot increases.
- Together with Buck, we've used KnotPlot to study the spectrum of random knotting in the *gordian regime*.

(look at random knots)

What other tools do we want?

- tools for topology, geometry, and algebra
- bias on knot theory and 3–manifolds
- minimal set: GeomView, KnotPlot, Knotscape, Maple, SnapPea, Surface Evolver
- source code available (except Maple)
- with a modest amount of work, all will run under Linux, Win32, SGI, and MacOSX (exception: GeomView under Win32)
- some interconnections already in place (GeomView/SnapPea, GeomView/Evolver, Knotscape/SnapPea)
- complete all the other connections
- have all these tools (and more) available at a screen touch
- new tools written for Colab's environment: what is a good data model for sharing mathematical objects?

SnapPea

- written by Jeff Weeks
- program to study and create hyperbolic 3-manifolds
- nicely written kernal in straight C (hurray!)
- best documented source code ever seen (except maybe for $\TeX$ and METAFONT)
- straightforward to write an interface to the SnapPea kernal
- now comes with a Python/Tk interface
- should be easy to implement a Maple interface
- KnotPlot interface coming soon...

Knotscape

- written by Jim Hoste and Morwen Thistlethwaite (with help from others)
- started as a interface to the Hoste/Thistlethwaite/Weeks database of knots, but now does much more using a tcl based interface
- Actions:
  - enter Dowker code
  - look up knot in table
  - PostScript output
  - compute knot polynomials
  - find homomorphisms
  - determinant and signature
  - hyperbolic invariants (uses SnapPea) such as volume, symmetries, horoball diagrams, canonical cell decomp., and more
- has a simple knot drawing tool (LinkSmith)
- right now only on Linux/UNIX, but should be easy to port to Win32

Surface Evolver

- written by Ken Brakke
- very flexible tool to study “surfaces” (implemented as simplicial complexes)
- excellent manual!
- widely used to study minimal surfaces and for knot energy minimization
- ambient space quite general
  - any dimension
  - collection of manifolds
- large collection of energy models
- easily extendible
- energy minimization for knots somewhat better than KnotPlot’s methods

GeomView

- written by Mark Phillips, Tamara Munzner, and others
- widely used by mathematicians and non-mathematicians
- 3D/4D general visualization tool from the Geometry Center
- large collection of packages and extensions
- can work in 3D elliptical, euclidean, or hyperbolic space
- interface to Surface Evolver
- Xlib/Motif (lesstif) based, limiting it to X-windows systems
- a Win32 port would be a lot of work, but perhaps not too bad

Various threads

- *CoLab* at the CECM lab at SFU (www.colab.sfu.ca): need to learn to use touch screens more effectively,
- More relevance to "real" knot tying, as done by the *IGKT* (*I*nternational *G*uild of *K*not *T*yers), climbing knots, fishing knots, hair braids, extension cords
- Make KnotPlot more user-friendly (especially for kids), a "*OELE*" (*O*pen *E*nded *L*earning *E*nvironment)
- Knot Server (www.colab.sfu.ca/KnotPlot/KnotServer)
  - a repository for knot images, models, and other data
  - uses JavaView (by Konrad Polthier) as a web based knot model viewer
  - ultimately expand to include all 1,701,936 knots up to 16 crossings (discovered by Thistlethwaite, Weeks, and Hoste in 1998)

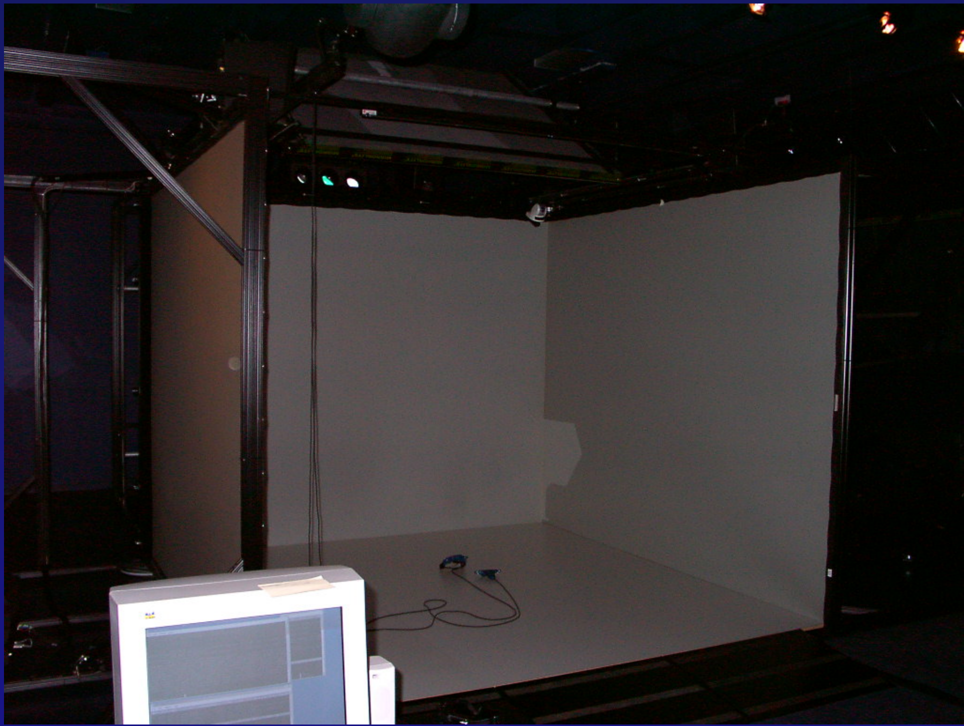
Extensions to KnotPlot

- caveKnotPlot — now working in the CAVE at NewMIC
  - potentially very useful to physical knot theory applications
    - surface of self contact for tight knots (of interest to polymer scientists and fishermen)
    - computational steering
- coKnotPlot — partially implemented
  - allow Maple (or any other program) to drive KnotPlot
  - allow one person's KnotPlot to drive someone else's KnotPlot
- integration of Darcy's tangle solver into KnotPlot (Isabel Darcy & Steve Levene, UTD)
- support plug in architecture (TOROS by Eric Rawdon)
- audKnotPlot — will use the simple OpenGL-like platform-independent audio library OpenAL

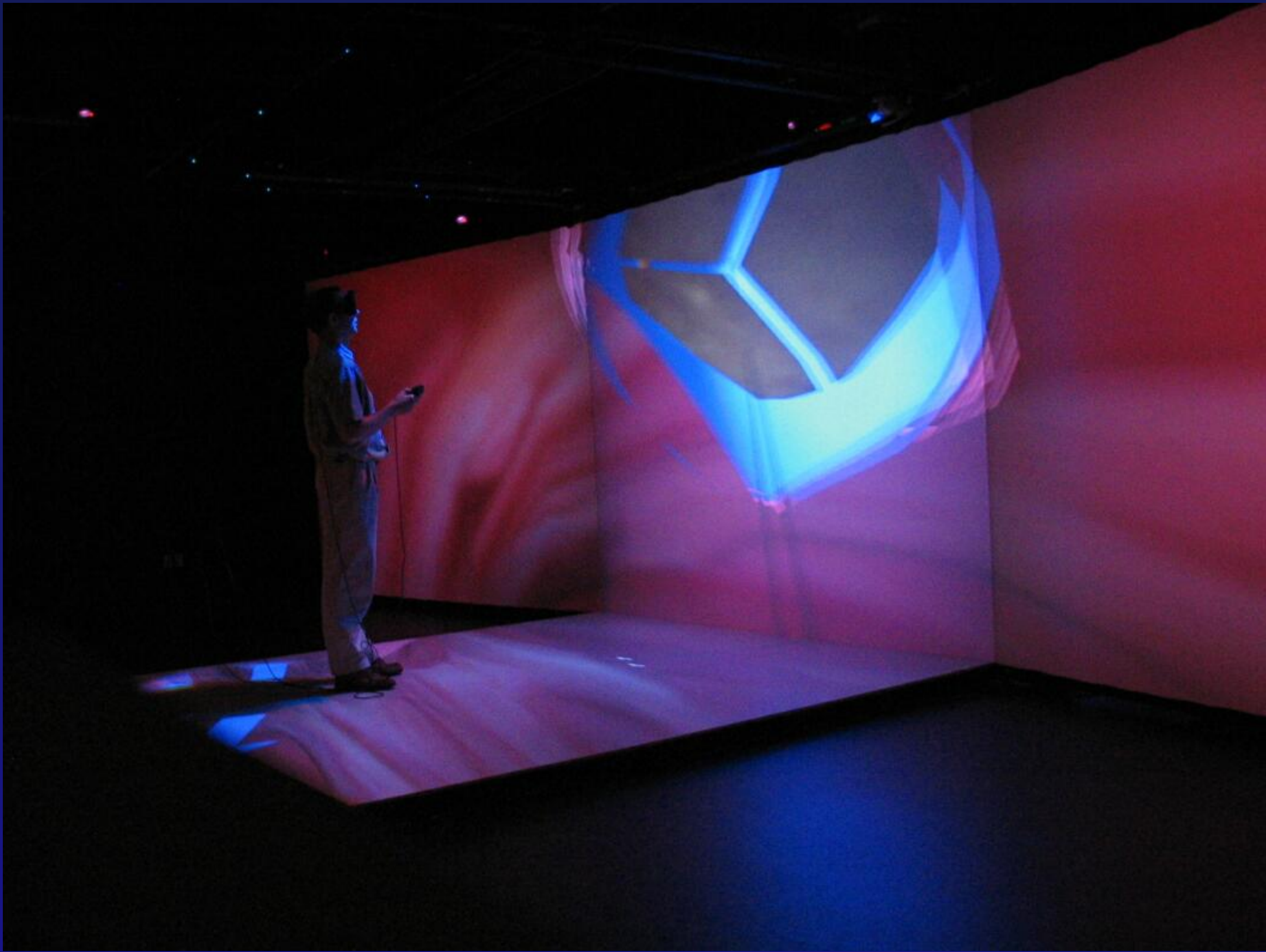

caveKnotPlot

caveKnotPlot

- KnotPlot in the CAVE environment at the Immersive Media Lab at NewMIC (Vancouver, Canada)



- CAVE is a 10ft$\times$10ft$\times$8ft “cube”



- IML operates in two main configurations, *cave*-mode, and *theatre*-mode (30ft wide wall)

IML in theatre mode with an audience

- caveKnotPlot uses CAVELib (TM)
- porting to CAVELib fairly easy for programs with simple data models
- software with dynamical data requires the programmer to worry about parallel programming issues (shared memory, semaphores, *etc.*)
- these problems are somewhat offset by packages such as IRIS Performer
- other packages exist (VR Juggler)
- generally fairly limited with regard to graphical user interface
- (demo)

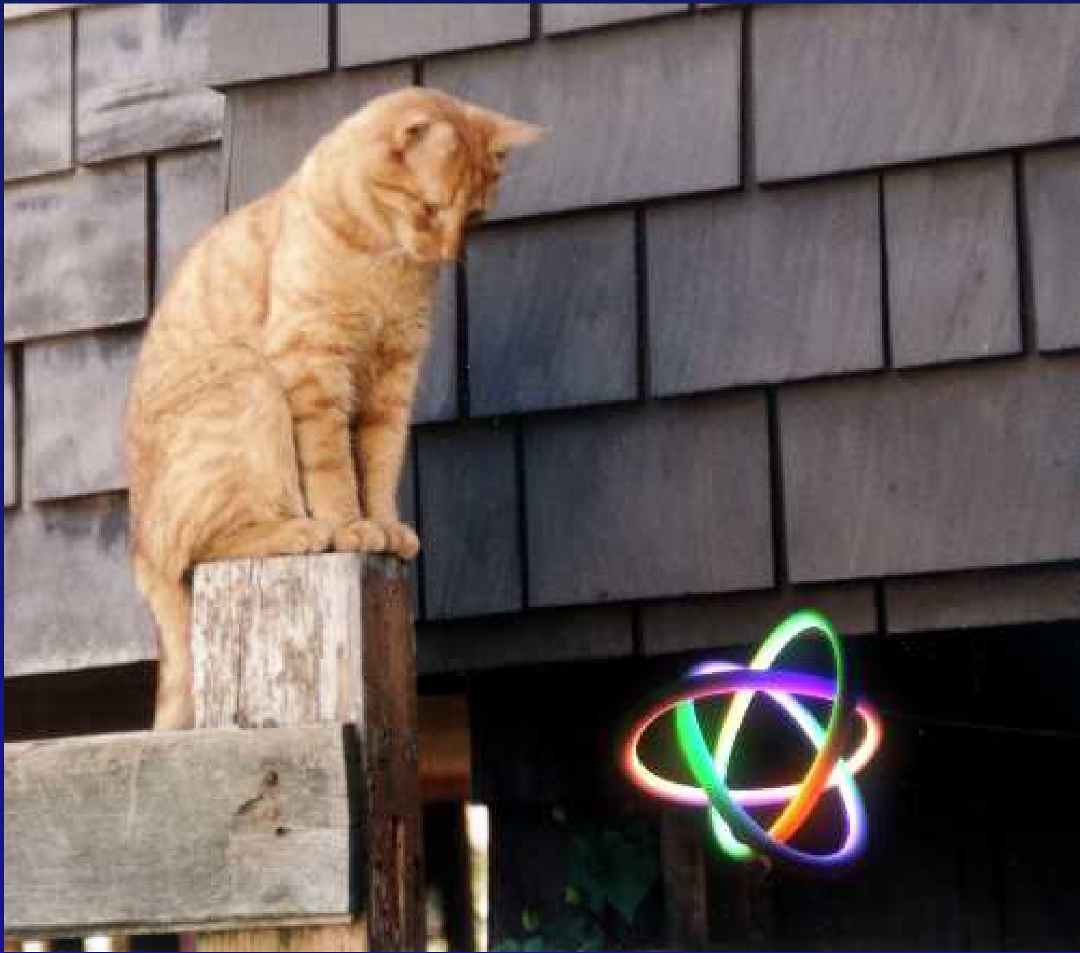
Other visualization applications in the IML

- Polyhedron visualizer (demo)
- Galaxy collider, based on seminal work by the brothers Toomre & Toomre in the 1970s and now a Linux screensaver (demo)
- KPUI (KnotPlot User Interface) can be used for general applications
- Special version of GLUT (OpenGL Utility Toolkit, written by Mark Kilgard) for cave like apps
- Port of Jeff Weeks' Curved Spaces to the CAVE (using CAVELib)
- Ultimately, everything should work in a cheaper laboratory environment or even in the "home cave"

coKnotPlot

- extension to KnotPlot to allow collaborative work
- required because knot theory typically brings together people from many different disciplines: mathematics, biology, physics, chemistry
- sharing of same data space, more than just the picture of what's going on
- currently operates in a master/slave mode, with ability to change roles
- uses TCP/IP for communication, should work between any pair of computers (two normal computers, computer to CAVE, CAVE to CAVE, CAVE to handheld device)
- will start exploring CAVE to CAVE use in next few months (imagine a "tug of war")

KnotPlot for math outreach

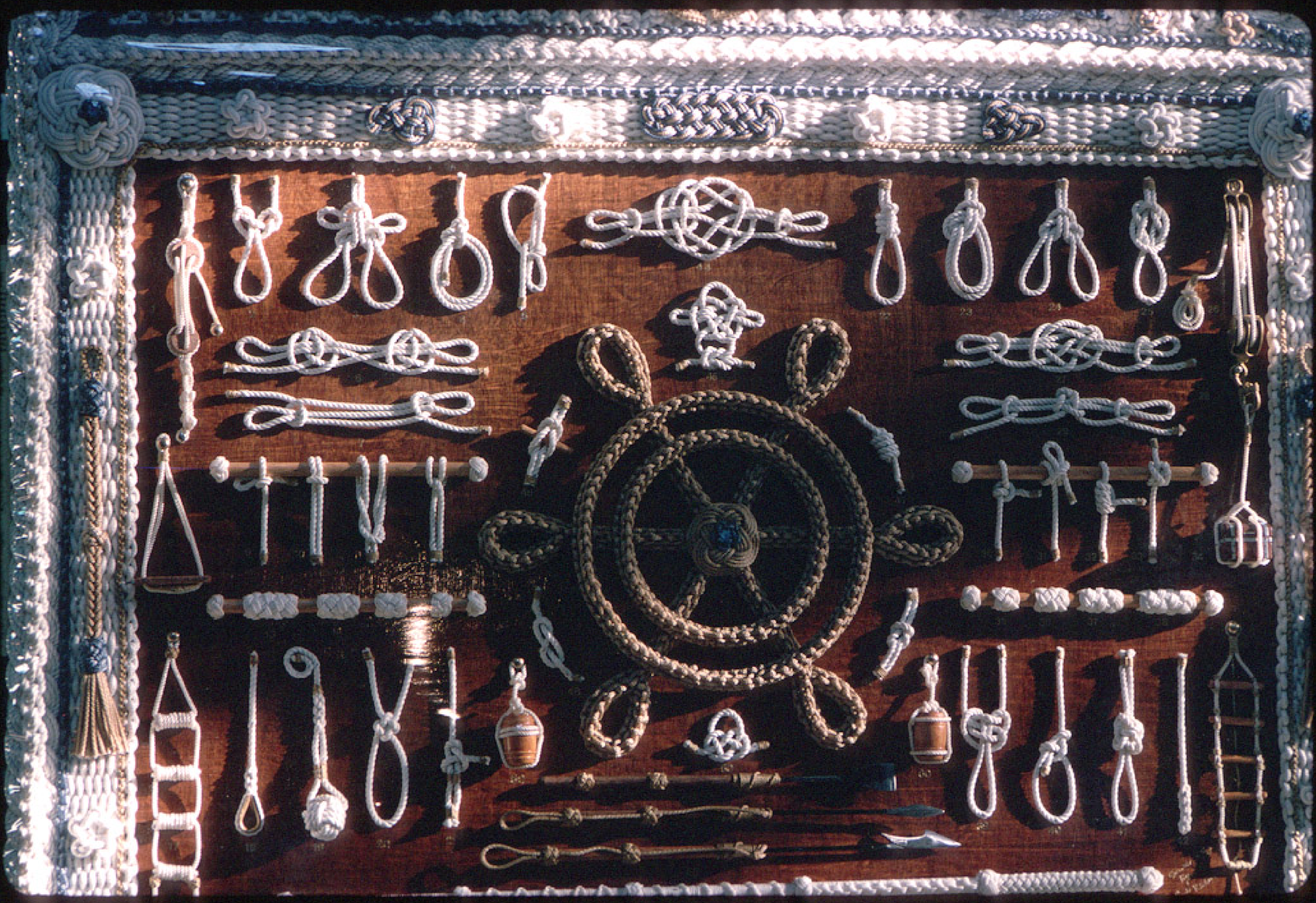


Hornby visualizing Borromean Rings

Lupin exploring some knots

The Knot Project

- The Knot Project is three knot theory researchers dedicated to bring knot theory to a large and diverse audience
  - Jonathan Simon — University of Iowa
  - Greg Buck — Saint Anselm College
  - Rob Scharein — wherever
- The Knot Project uses widespread cultural familiarity with knotting, braiding, and weaving to introduce beautiful and scientifically important mathematics to people who might otherwise not be open to thinking mathematically.



Goals of the exhibit

Why Knots?

- knots are fun
- knots are also mathematics
- knots are a bridge between the cultural and scientific ways of knowing
- knots are ubiquitous

Knot theory has advantages over other branches of mathematics

- knots are accessible mathematics
- knots are tactile and visual

Exhibit plans

- Many examples of knots in all areas where the tangling of filaments is important
- Interactive computer-based activity areas. (activities)
- Large projection screen with non-stop, non-repeating, non-terminating randomized KnotPlot graphics.
- CD-ROM: visitors can take everything home with them.

Why this will work

- Knot Project has a considerable amount of combined experience bringing knots to non-mathematicians.
- GB has taught pre-schoolers how to braid: most are amazed that you can add something that isn't a number.
- KnotPlot not an game, but a creative passtime: this appeals to children.
- Knot Project (GB and RS) meeting with the International Guild of Knot Tyers at the New Bedford Whaling Museum in August 1997.
- Knot Project gave a presentation at the Association of Science and Technology Centers in Edmonton in October 1998:
  - garnered a great deal of interest from science centres, and other organizations involved with Science and Math outreach
  - backed by the Girl Scouts of America.

Saskatchewan (sp?) in January

Gordon Snelgrove Art Gallery
University of Saskatchewan
January 7–18, 2002




The big knot in the centre of the gallery was a simple trefoil made of 50 linear feet of 20 inch diameter flexhaust.

Knot Exhibit at the Museum of Science, Boston

- opening Summer 2003
- part of a much larger permanent exhibit on models
- knot artifacts and tactile objects designed by Greg Buck
- electronic component using KnotPlot:
  - runs in continuous demo mode, as an "attractor"
  - can be interrupted at any time for interactive exploration: draw and relax a knot, explore a database of cool knots
  - must be completely robust (both as software and also physically)
- designed to be separate from rest of exhibit, for easy porting to other science museums

For more....

- KnotPlot Site

  www.knotplot.com

  for a lot's of pictures, movies, and a free copy of the software (for Windows 9x/NT/ME/2000/XP, Linux, Solaris, IRIX, Mac OSX)

- KnotPlot Research and Development Site

  www.colab.sfu.ca/KnotPlot

- Email

  scharein@cecm.sfu.ca
  rob@knots.org